Proof for the equivalence of local and global convexities in section 2.1

As in the paper, let Δ be a d-dimensional triangulation with point configuration $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n$. Let $w_1, \ldots, w_n$ be its lifting. Let L be the graph of the lifting:

$$L = \bigcup_{\boldsymbol{p}_{i_0}, \ldots, \boldsymbol{p}_{i_d} : d\text{-simplex of } \Delta} \operatorname{conv}(\begin{pmatrix} \boldsymbol{p}_{i_0} \\ w_{i_0} \end{pmatrix}, \ldots, \begin{pmatrix} \boldsymbol{p}_{i_d} \\ w_{i_d} \end{pmatrix}).$$

The local convexity and global convexity is equivalent when $d = 1$. For $d > 1$, suppose a lifting was locally convex, but not globally convex.

1. Since the lifting is not globally convex, we can take $\boldsymbol{q}, \boldsymbol{r} \in L$, and $\boldsymbol{s}$ from (the relative interior of) the line segment $\operatorname{conv}(\boldsymbol{q}, \boldsymbol{r})$ such that $\boldsymbol{s}$ is below L with respect to the x_{d+1} axis. This can be done, for example, by choosing a d-simplex $\operatorname{conv}(\boldsymbol{p}_{i_0}, \ldots, \boldsymbol{p}_{i_d})$ in Δ and a point $\boldsymbol{p}_j \notin \{\boldsymbol{p}_{i_0}, \ldots, \boldsymbol{p}_{i_d}\}$ violating the criterion for global convexity and defining

$$\begin{aligned} \boldsymbol{q} &= \frac{1}{d+1} \left(\begin{pmatrix} \boldsymbol{p}_{i_0} \\ w_{i_0} \end{pmatrix} + \cdots + \begin{pmatrix} \boldsymbol{p}_{i_d} \\ w_{i_d} \end{pmatrix} \right), \\ \boldsymbol{r} &= \begin{pmatrix} \boldsymbol{p}_j \\ w_j \end{pmatrix}, \\ \boldsymbol{s} &= \epsilon \boldsymbol{q} + (1 - \epsilon) \boldsymbol{r}, \end{aligned}$$

   for $\epsilon > 0$ small enough.

2. Let the plane including $\boldsymbol{q}, \boldsymbol{r}, \boldsymbol{s}$ and parallel to the x_{d+1} axis be π. In other words, $\pi = \operatorname{aff}(\boldsymbol{q}, \boldsymbol{r}, \boldsymbol{s}, \boldsymbol{q} + \boldsymbol{e}_{d+1})$, where $\boldsymbol{e}_{d+1}$ is the $(d+1)$-th unit vector.

3. The intersection

$$\Delta \cap \pi = \{\sigma \cap \pi : \sigma \in \Delta\}$$

   is a one-dimensional triangulation, and $L \cap \pi$ is its graph (for some lifting). The local convexity of the lifting of Δ implies the local convexity of the lifting of $\Delta \cap \pi$. Points $\boldsymbol{q}, \boldsymbol{r}, \boldsymbol{s}$ show that this lifting is not globally convex. This contradicts the equivalence of the local and global convexities for the one-dimensional case.