Nonregular triangulations, view graphs of triangulations, and linear programming duality

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Abstract

For a triangulation and a point, we define a directed graph representing the order of the maximal dimensional simplices in the triangulation viewed from the point. We prove that triangulations having a cycle the reverse of which is not a cycle in this graph viewed from some point are forming a (proper) subclass of nonregular triangulations. We use linear programming duality to investigate further properties of nonregular triangulations in connection with this graph.

1 Introduction

Let $\mathcal{A} = \{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n\} \subset \mathbb{R}^d$ be a point configuration with its convex hull $\mathrm{conv}(\mathcal{A})$ being a d-polytope. A *triangulation* Δ *of* $\mathcal{A}$ is a geometric simplicial complex with its vertices among $\mathcal{A}$ and the union of its faces equal to $\mathrm{conv}(\mathcal{A})$. A triangulation is *regular* (or *coherent*) if it can appear as the projection of the lower boundary of a $(d+1)$-polytope in $\mathbb{R}^{d+1}$. If not, the triangulation is *nonregular*.

Starting from the study of generalized hypergeometric functions, Gel'fand, Kapranov & Zelevinskiĭ showed that regular triangulations altogether of a point configuration are forming a polytopal structure described by the secondary polytope [4] [5]. In connection to Gröbner bases, Sturmfels showed that initial ideals for the affine toric ideal determined by a point configuration correspond to the regular triangulations of the point configuration [8] [9]. Regular triangulations are a generalization of the Delaunay triangulation well known in computational geometry, and have also been used extensively in this field [2].

Though nonregular triangulations are know to be behaving differently from regular triangulations, they are not well understood yet. Santos showed a nonregular triangulation with no flips indicating that a flip graph can be disconnected, which never happens when restricted to regular triangulations [7]. Ohsugi & Hibi showed the existence of a point configuration with no unimodular regular triangulations, but with a unimodular nonregular triangulation [6]. Also, de Loera, Hoşten, Santos & Sturmfels showed that cyclic polytopes can have exponential number of nonregular triangulations compared to polynomial number of regular ones [1]. The aim of this paper is to put some insight into nonregular triangulations.

Hereafter in this paper, we fix a triangulation Δ. For the triangulation Δ and a point $\boldsymbol{v}$ in $\mathbb{R}^d$, we define the *graph* $G_{\boldsymbol{v}}$ *of* Δ *viewed from* $\boldsymbol{v}$ as the graph with its vertices corresponding to the d-simplices of Δ and a directed edge $\overrightarrow{\sigma\tau}$ existing when $\boldsymbol{v}$ belongs to the closed halfspace having the affine hull $\mathrm{aff}(\sigma \cap \tau)$ as its boundary and including σ. When $\boldsymbol{v} \in \mathrm{aff}(\sigma \cap \tau)$, both edges $\overrightarrow{\sigma\tau}, \overrightarrow{\tau\sigma}$ appear in $G_{\boldsymbol{v}}$. The graph $G_{\boldsymbol{v}}$ is a directed graph with the underlying undirected graph the adjacency graph of the d-simplices in Δ. Of course, $G_{\boldsymbol{v}}$ might differ for different choices of $\boldsymbol{v}$. Though there are infinite choices of viewpoints $\boldsymbol{v}$, there are only finitely many possibilities of view graphs $G_{\boldsymbol{v}}$.

A sequence of vertices $\sigma_1, \sigma_2, \ldots, \sigma_i, \sigma_1$ in $G_{\boldsymbol{v}}$ forms a *cycle* when $\overrightarrow{\sigma_1\sigma_2}, \ldots, \overrightarrow{\sigma_{i-1}\sigma_i}$, $\overrightarrow{\sigma_i\sigma_1}$ are edges of $G_{\boldsymbol{v}}$ and $\sigma_i \neq \sigma_j$ for $i \neq j$. We define a cycle $\sigma_1, \sigma_2, \ldots, \sigma_i, \sigma_1$ to be *contradicting* when the reverse order $\sigma_1, \sigma_i, \ldots, \sigma_2, \sigma_1$ is not a cycle in $G_{\boldsymbol{v}}$. For vertices $\sigma_1, \ldots, \sigma_i$ in $G_{\boldsymbol{v}}$, edges $\overrightarrow{\sigma_1\sigma_2}, \ldots, \overrightarrow{\sigma_{i-1}\sigma_i}, \overrightarrow{\sigma_2\sigma_1}, \ldots, \overrightarrow{\sigma_i\sigma_{i-1}}$ exist if and only if $\boldsymbol{v} \in \mathrm{aff}(\sigma_1 \cap \cdots \cap \sigma_i)$.

Regularity of a triangulation can be stated as a linear programming problem, so the two subjects obviously have connection. But, an interesting point in our argument is that we use linear programming duality to analyze further in detail some properties of nonregular triangulations.

For any triangulation, the condition of regularity can be written as a linear programming problem. The variables $w_1, \ldots, w_n$ correspond to the lifting (or weight) of the vertices $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n$. The inequality constraints correspond to the interior $(d-1)$-simplices in Δ and describes the local convexity of the two d-simplices intersecting there. Altogether, we get a system of inequalities $A\boldsymbol{w} > \boldsymbol{0}$ ($\boldsymbol{0}$ is the zero vector), and the triangulation is regular when this has a solution. Easily, this is equivalent to $A\boldsymbol{w} \geq \boldsymbol{1}$ ($\boldsymbol{1}$ is the vector with all entries one) having a solution. By linear programming duality (or Farkas' lemma), the triangulation is nonregular if and only if the *dual problem* $\boldsymbol{u}A = \boldsymbol{0}$, $\boldsymbol{u} \geq \boldsymbol{0}$ has a nonzero solution.

Our main theorem constructs a nonzero solution of the dual problem combinatorially and explicitly from a contradicting cycle.

Theorem. *For a triangulation Δ, if a graph $G_{\boldsymbol{v}}$ viewed from some point $\boldsymbol{v}$ contains a contradicting cycle, in correspondence with this cycle, we can make a nonzero solution of the dual problem stated above. Thus, Δ is nonregular. The support set* (*i.e. collection of nonzero elements*) *of this solution is a subset of the edges forming the cycle. On the other hand, the reverse of the claim above is not true. There exists a nonregular triangulation with none of its view graphs $G_{\boldsymbol{v}}$ containing a contradicting cycle.* (*See Example* 3.3)

The theorem says that triangulations containing a contradicting cycle in its graph $G_{\boldsymbol{v}}$ viewed from some point $\boldsymbol{v}$ are forming a (proper) subclass of nonregular triangulations. This subclass of triangulations is interesting in that they have combinatorial explanation. On the other hand, regularity or nonregularity, defined by linear inequalities, are of continuous nature. This is the first attempt to give a (combinatorial) subclass of nonregular triangulations. Even if we consider contradicting closed paths instead of contradicting cycles, allowing to pass the same vertex more than once, the class of the triangulations having such contradicting thing in its view graph does not change, because any contradicting closed path includes a contradicting cycle.

Checking that Example 3.3 is a counterexample to the reverse of the implication in the theorem (i.e. the view graph from any viewpoint does not contain a contradicting cycle), can be done by extensive enumeration of view graphs. However, by describing nonregularity as a linear programming problem, and using linear programming duality, we prove the counterexample in a more elegant way.

A similar but different directed graph of a triangulation viewed from a point has been studied by Edelsbrunner [3]. This was in the context of computer vision, and his graph represents the in_front/behind relation among simplices of any dimension, even not adjacent to each other. When our graph and the restriction of Edelsbrunner's graph to d-simplices are compared, neither includes the other in general. However, if we take the transitive closure of our graph, it includes his graph as a subgraph (possibly with more edges). Our graph might be more appropriate in describing combinatorial structures of triangulations, because their underlying undirected graphs are the adjacency graph of d-simplices. Edelsbrunner proves that if a triangulation is regular, his graph viewed from any point is "acyclic". The line shelling argument in a note there gives a proof for the contrapositive of our theorem, but without explicit constructioin of a solution of the dual problem.

2 Regularity, linear programming, and duality

2.1 Inequalities for regularity

A triangulation Δ of the point configuration $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n$ is regular if their exists a lifting (or weight) $w_1, \ldots, w_n \in \mathbb{R}$ such that the projection of the lower boundary with respect to the x_{d+1} axis of the $(d+1)$-polytope $\mathrm{conv}(\binom{\boldsymbol{p}_1}{w_1}, \ldots, \binom{\boldsymbol{p}_n}{w_n})$ becomes Δ. This condition can be described by inequalities with $w_1, \ldots, w_n$ the variables.

A straightforward description of this "global" convexity is as follows:

- For each d-simplex $\mathrm{conv}(\boldsymbol{p}_{i_0}, \ldots, \boldsymbol{p}_{i_d})$ in Δ, and any point $\boldsymbol{p}_j \notin \{\boldsymbol{p}_{i_0}, \ldots, \boldsymbol{p}_{i_d}\}$, the lifted point $\binom{\boldsymbol{p}_j}{w_j}$ is above the hyperplane $\mathrm{aff}(\binom{\boldsymbol{p}_{i_0}}{w_{i_0}}, \ldots, \binom{\boldsymbol{p}_{i_d}}{w_{i_d}})$ in $\mathbb{R}^{d+1}$:

$$\begin{vmatrix} 1 & \cdots & 1 \\ \boldsymbol{p}_{i_0} & \cdots & \boldsymbol{p}_{i_d} \end{vmatrix} \begin{vmatrix} 1 & \cdots & 1 & 1 \\ \boldsymbol{p}_{i_0} & \cdots & \boldsymbol{p}_{i_d} & \boldsymbol{p}_j \\ w_{i_0} & \cdots & w_{i_d} & w_j \end{vmatrix} > 0.$$

However, the above condition is equivalent to the following "local" convexity, with much less inequalities:

- For each interior $(d-1)$-simplex $\mathrm{conv}(\boldsymbol{p}_{i_1}, \ldots, \boldsymbol{p}_{i_d})$ in Δ, where the two d-simplices $\mathrm{conv}(\boldsymbol{p}_{i_0}, \boldsymbol{p}_{i_1}, \ldots, \boldsymbol{p}_{i_d})$ and $\mathrm{conv}(\boldsymbol{p}_{i_1}, \ldots, \boldsymbol{p}_{i_d}, \boldsymbol{p}_{i_{d+1}})$ are intersecting, the lifted point $\binom{\boldsymbol{p}_{i_{d+1}}}{w_{i_{d+1}}}$ is above the hyperplane $\mathrm{aff}(\binom{\boldsymbol{p}_{i_0}}{w_{i_0}}, \ldots, \binom{\boldsymbol{p}_{i_d}}{w_{i_d}})$ in $\mathbb{R}^{d+1}$:

$$\begin{vmatrix} 1 & \cdots & 1 \\ \boldsymbol{p}_{i_0} & \cdots & \boldsymbol{p}_{i_d} \end{vmatrix} \begin{vmatrix} 1 & \cdots & 1 & 1 \\ \boldsymbol{p}_{i_0} & \cdots & \boldsymbol{p}_{i_d} & \boldsymbol{p}_{i_{d+1}} \\ w_{i_0} & \cdots & w_{i_d} & w_{i_{d+1}} \end{vmatrix} > 0. \qquad (*)$$

The equivalence of these two convexity conditions is proved easily by reducing to the one dimensional case.

We define the collection of the inequalities $(*)$ for all interior $(d-1)$-simplices in Δ as

$$A\boldsymbol{w} > \boldsymbol{0}.$$

We denote this *matrix representing the regularity of* Δ by A.

Lemma 2.1. *For a triangulation* Δ, *and the matrix* A *representing its regularity, we have*

$$\begin{aligned} &\Delta \text{ is regular} \\ \Leftrightarrow\ &\text{there exists } \boldsymbol{w} \in \mathbb{R}^n,\ A\boldsymbol{w} > \boldsymbol{0}, \\ \Leftrightarrow\ &\text{there exists } \boldsymbol{w} \in \mathbb{R}^n,\ A\boldsymbol{w} \geq \boldsymbol{1}. \end{aligned}$$

By linear programming duality (or Farkas' lemma), we have

$$\begin{aligned} &\Delta \text{ is nonregular} \\ \Leftrightarrow\ &\text{there does not exist } \boldsymbol{w} \in \mathbb{R}^n,\ A\boldsymbol{w} \geq \boldsymbol{1}, \\ \Leftrightarrow\ &\text{there exists } \boldsymbol{u} \geq \boldsymbol{0},\ \boldsymbol{u}A = \boldsymbol{0},\ \boldsymbol{u} \neq \boldsymbol{0}. \end{aligned}$$

Thus, the (non)regularity of Δ can be judged by the existence of a nonzero point in the polyhedron $\{\boldsymbol{u} \geq \boldsymbol{0} : \boldsymbol{u}A = \boldsymbol{0}\}$ of the set of solutions of the dual problem.

2.2 A nonzero solution of the dual problem from a contradicting cycle

Here, we give an explicit construction of a nonzero solution of the dual problem $\boldsymbol{u}A = \boldsymbol{0}, \boldsymbol{u} \geq \boldsymbol{0}$, from a contradicting cycle in the graph $G_{\boldsymbol{v}}$ viewed from some point $\boldsymbol{v}$. For $\boldsymbol{v} \in \mathbb{R}^d$, a d-simplex σ in Δ, and $\boldsymbol{w} \in \mathbb{R}^n$, we let

$$\begin{aligned} x_{d+1}(\boldsymbol{v}, \sigma, \boldsymbol{w}) = \ &(\text{the } x_{d+1} \text{ coordinate of the point} \\ &(\text{the hyperplane containing the lifting of the } d\text{-simplex } \sigma \text{ by } \boldsymbol{w}) \\ &\cap \{(\boldsymbol{v}, x_{d+1}) : x_{d+1} \in \mathbb{R}\}). \end{aligned}$$

Lemma 2.2. *Let* Δ *be a triangulation,* A *the matrix representing its regularity, and* $\boldsymbol{v} \in \mathbb{R}^d$. *For an edge* $\overrightarrow{\sigma\tau}$ *in the graph* $G_{\boldsymbol{v}}$ *viewed from* $\boldsymbol{v}$, *there exists a constant* $\alpha_{\sigma\cap\tau} \geq 0$ *such that*

$$x_{d+1}(\boldsymbol{v}, \sigma, \boldsymbol{w}) - x_{d+1}(\boldsymbol{v}, \tau, \boldsymbol{w}) = \alpha_{\sigma\cap\tau} A_{\sigma\cap\tau,*}\boldsymbol{w} \qquad (\text{for any } \boldsymbol{w} \in \mathbb{R}^n),$$

where $A_{\sigma\cap\tau,*}$ *is the row of* A *corresponding to the interior* $(d-1)$*-simplex* $\sigma \cap \tau$ *in* Δ. *Furthermore,* $\boldsymbol{v} \in \operatorname{aff}(\sigma \cap \tau)$ *if and only if* $\alpha_{\sigma\cap\tau} = 0$.

Proof. Straightforward. □

Now we construct a nonzero solution of the dual problem from a contradicting cycle. This gives the proof of our main theorem.

Proof. **(main theorem)** Suppose we have a contradicting cycle $\sigma_1, \sigma_2, \ldots, \sigma_i, \sigma_1$ in $G_{\boldsymbol{v}}$. By Lemma 2.2, we can find $\alpha_{\sigma_1\cap\sigma_2}, \ldots, \alpha_{\sigma_i\cap\sigma_1} \geq 0$, or their collection as a vector

$\boldsymbol{\alpha} \geq \mathbf{0}$, satisfying for any $\boldsymbol{w} \in \mathbb{R}^n$,

$$
\begin{aligned}
& x_{d+1}(\boldsymbol{v}, \sigma_1, \boldsymbol{w}) - x_{d+1}(\boldsymbol{v}, \sigma_2, \boldsymbol{w}) \\
& \quad \cdots \\
& \quad + x_{d+1}(\boldsymbol{v}, \sigma_i, \boldsymbol{w}) - x_{d+1}(\boldsymbol{v}, \sigma_1, \boldsymbol{w}) \\
& = \alpha_{\sigma_1 \cap \sigma_2} A_{\sigma_1 \cap \sigma_2, *} \boldsymbol{w} \\
& \quad \cdots \\
& \quad + \alpha_{\sigma_i \cap \sigma_1} A_{\sigma_i \cap \sigma_1, *} \boldsymbol{w} \\
& = \boldsymbol{\alpha} A \boldsymbol{w} \quad (\boldsymbol{\alpha} \text{ is a vector with those elements not in the cycle } 0) \\
& = 0.
\end{aligned}
$$

Thus, $\boldsymbol{\alpha} A = \mathbf{0}$. Since we took a contradicting cycle, by Lemma 2.2, $\boldsymbol{\alpha} \neq \mathbf{0}$. Hence, we obtain a nonzero solution of the dual problem $\boldsymbol{u} A = \mathbf{0}, \boldsymbol{u} \geq \mathbf{0}$. This together with Lemma 2.1 proves the claim of the main theorem. □

2.3 Recognizing nonregularity or finding contradicting cycles

Judging whether the given triangulation Δ is (non)regular reduces to judging whether the inequalities $A\boldsymbol{w} \geq \mathbf{1}$, with A the matrix of regularity, has a solution $\boldsymbol{w}$. This is a linear programming problem, and can be computed, for example by interior point method, in polynomial time.

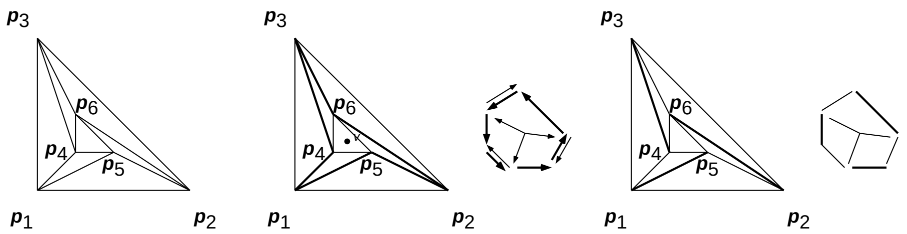
One way to judge if a triangulation Δ has a contradicting cycle in some view graph $G_{\boldsymbol{v}}$ is to enumerate all possible view graphs and enumerate the cycles there. The generation of view graphs can be done, for example, by generating all graphs viewed from the minimal cells in the hyperplane arrangement made by the affine hulls of the interior $(d-1)$-simplices in Δ.

3 Examples

Example 3.1 (A nonregular triangulation with 6 vertices). For the point configuration

$$\boldsymbol{p}_1 = (0\,0), \qquad \boldsymbol{p}_2 = (4\,0), \qquad \boldsymbol{p}_3 = (0\,4),$$
$$\boldsymbol{p}_4 = (1\,1), \qquad \boldsymbol{p}_5 = (2\,1), \qquad \boldsymbol{p}_6 = (1\,2),$$

we consider the triangulation Δ indicated in Figure 1(a) below. The graph $G_{\boldsymbol{v}}$ viewed from $\boldsymbol{v} = (\frac{4}{3}\,\frac{4}{3})$ is in Figure 1(b). It has one contradicting cycle $\boldsymbol{p}_1\boldsymbol{p}_4\boldsymbol{p}_5, \boldsymbol{p}_1\boldsymbol{p}_2\boldsymbol{p}_5, \boldsymbol{p}_2\boldsymbol{p}_5\boldsymbol{p}_6, \boldsymbol{p}_2\boldsymbol{p}_3\boldsymbol{p}_6, \boldsymbol{p}_3\boldsymbol{p}_4\boldsymbol{p}_6, \boldsymbol{p}_1\boldsymbol{p}_3\boldsymbol{p}_4, \boldsymbol{p}_1\boldsymbol{p}_4\boldsymbol{p}_5$ denoted by bold edges. The matrix representing the regularity of Δ is



(a) triangulation (b) view graph from $\boldsymbol{v}$ (c) support of the dual solutions

Figure 1: Example 3.1.

$A =$

	w_1	w_2	w_3	w_4	w_5	w_6
$\boldsymbol{p}_1\boldsymbol{p}_4$	3		1	-8	4	
$\boldsymbol{p}_1\boldsymbol{p}_5$	-1	1		4	-4	
$\boldsymbol{p}_2\boldsymbol{p}_5$	1	3			-8	4
$\boldsymbol{p}_2\boldsymbol{p}_6$		-1	1		4	-4
$\boldsymbol{p}_3\boldsymbol{p}_4$	1		-1	-4		4
$\boldsymbol{p}_3\boldsymbol{p}_6$		1	3	4		-8
$\boldsymbol{p}_4\boldsymbol{p}_5$	1			-3	1	1
$\boldsymbol{p}_4\boldsymbol{p}_6$			1	1	1	-3
$\boldsymbol{p}_5\boldsymbol{p}_6$		1		1	-3	1

.

The polyhedron of the solutions of the dual problem is

$$\{\boldsymbol{u} \geq \boldsymbol{0} : A\boldsymbol{u} = \boldsymbol{0}\} = \mathbb{R}_{\geq 0}(0\,1\,0\,1\,1\,0\,0\,0\,0),$$

where interior 1-simplices are indexed lexicographically. The support of the nonzero solutions is denote by bold edges in Figure 1(c). Remark that they are included in the (underlying undirected) edges of the contradicting cycle.

Example 3.2 (Another nonregular triangulation with 6 vertices). The vertex $\boldsymbol{p}_2$ in Examples 3.1 is perturbed. The point configuration is

$$\boldsymbol{p}_1 = (0\,0), \qquad \boldsymbol{p}_2 = (\tfrac{7}{2}\,0), \qquad \boldsymbol{p}_3 = (0\,4),$$
$$\boldsymbol{p}_4 = (1\,1), \qquad \boldsymbol{p}_5 = (2\,1), \qquad \boldsymbol{p}_6 = (1\,2).$$

The triangulation Δ is indicated in Figure 2 below. Each of the graph viewed from $\boldsymbol{v}_1 = (\frac{5}{4}\,\frac{3}{2})$, $\boldsymbol{v}_2 = (\frac{4}{3}\,\frac{4}{3})$, or $\boldsymbol{v}_3 = (\frac{7}{5}\,\frac{7}{5})$ has a unique contradicting cycle. The matrix

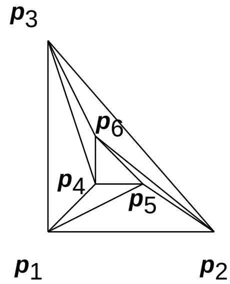


Figure 2: Triangulation of Example 3.2.

representing the regularity of Δ is

$$A = \begin{array}{c|cccccc}
 & w_1 & w_2 & w_3 & w_4 & w_5 & w_6 \\ \hline
\boldsymbol{p}_1\boldsymbol{p}_4 & 3 & & 1 & -8 & 4 & \\
\boldsymbol{p}_1\boldsymbol{p}_5 & -1 & 1 & & \frac{7}{2} & -\frac{7}{2} & \\
\boldsymbol{p}_2\boldsymbol{p}_5 & \frac{1}{2} & 3 & & & -7 & \frac{7}{2} \\
\boldsymbol{p}_2\boldsymbol{p}_6 & & -1 & \frac{1}{2} & & 3 & -\frac{5}{2} \\
\boldsymbol{p}_3\boldsymbol{p}_4 & 1 & & -1 & -4 & & 4 \\
\boldsymbol{p}_3\boldsymbol{p}_6 & & 1 & \frac{5}{2} & 3 & & -\frac{13}{2} \\
\boldsymbol{p}_4\boldsymbol{p}_5 & 1 & & & -3 & 1 & 1 \\
\boldsymbol{p}_4\boldsymbol{p}_6 & & & 1 & 1 & 1 & -3 \\
\boldsymbol{p}_5\boldsymbol{p}_6 & & 1 & & \frac{1}{2} & -\frac{5}{2} & 1
\end{array}.$$

The polyhedron of the solutions of the dual problem is

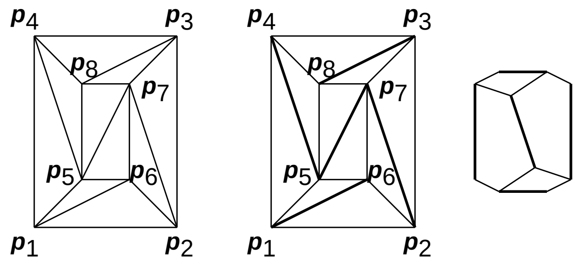
$$\begin{aligned}
&\{\boldsymbol{u} \geq \boldsymbol{0} : A\boldsymbol{u} = \boldsymbol{0}\} \\
&= \mathbb{R}_{\geq 0}(1\,8\,0\,8\,5\,0\,0\,0\,0) \\
&\quad + \mathbb{R}_{\geq 0}(0\,8\,2\,14\,7\,0\,0\,0\,0) \\
&\quad + \mathbb{R}_{\geq 0}(0\,6\,0\,7\,6\,1\,0\,0\,0) \\
&\quad + \mathbb{R}_{\geq 0}(0\,2\,0\,2\,1\,0\,1\,0\,0) \\
&\quad + \mathbb{R}_{\geq 0}(0\,2\,0\,2\,2\,0\,0\,1\,0) \\
&\quad + \mathbb{R}_{\geq 0}(0\,1\,0\,2\,1\,0\,0\,0\,1),
\end{aligned}$$

where interior 1-simplices are indexed lexicographically. The first three 1-rays correspond to the solutions made by the contradicting cycles in view graphs $G_{\boldsymbol{v}_1}, G_{\boldsymbol{v}_2}, G_{\boldsymbol{v}_3}$, as in Subsection 2.2. The latter three 1-rays have no such correspondence.

Example 3.3 (Counterexample to the reverse of the main theorem). With the point configuration

$$\boldsymbol{p}_1 = (0\,0), \quad \boldsymbol{p}_2 = (3\,0), \quad \boldsymbol{p}_3 = (3\,4), \quad \boldsymbol{p}_4 = (0\,4),$$
$$\boldsymbol{p}_5 = (1\,1), \quad \boldsymbol{p}_6 = (2\,1), \quad \boldsymbol{p}_7 = (2\,3), \quad \boldsymbol{p}_8 = (1\,3),$$

the triangulation Δ indicated in Figure 3(a) below is a nonregular triangulation with none of its view graphs $G_{\boldsymbol{v}}$ containing a contradicting cycle. The matrix representing



(a) triangulation (b) support of the dual solutions

Figure 3: Example 3.3.

the regularity of Δ is

$A =$

	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8
$\boldsymbol{p}_1\boldsymbol{p}_5$	3			1	−8	4		
$\boldsymbol{p}_1\boldsymbol{p}_6$	−1	1			3	−3		
$\boldsymbol{p}_2\boldsymbol{p}_6$	2	4				−9	3	
$\boldsymbol{p}_2\boldsymbol{p}_7$		−2	2			4	−4	
$\boldsymbol{p}_3\boldsymbol{p}_7$		1	3				−8	4
$\boldsymbol{p}_3\boldsymbol{p}_8$			−1	1			3	−3
$\boldsymbol{p}_4\boldsymbol{p}_8$			2	4	3			−9
$\boldsymbol{p}_4\boldsymbol{p}_5$	2			−2	−4			4
$\boldsymbol{p}_5\boldsymbol{p}_6$	2				−4	1	1	
$\boldsymbol{p}_6\boldsymbol{p}_7$		2			2	−5	1	
$\boldsymbol{p}_7\boldsymbol{p}_8$			2		1		−4	1
$\boldsymbol{p}_5\boldsymbol{p}_8$				2	1		2	−5
$\boldsymbol{p}_5\boldsymbol{p}_7$					−2	2	−2	2

The polyhedron of the solutions of the dual problem is

$$\{\boldsymbol{u} \geq \boldsymbol{0} : A\boldsymbol{u} = \boldsymbol{0}\} = \mathbb{R}_{\geq 0}(0\,2\,0\,1\,0\,2\,0\,1\,0\,0\,0\,0\,1),$$

where interior 1-simplices are indexed lexicographically. The support of the nonzero solutions is denote by bold edges in Figure 3(b). If a contradicting cycle existed for some view graph $G_{\boldsymbol{v}}$, this (directed) cycle should contain all of the bold edges (in its underlying undirected counterpart). However, there are no cycles containing all of these bold edges. Hence, there exists no view graph G_v containing a contradicting cycle for this example. (Remark: If we take the edge p_6p_8 instead of p_5p_7, this new flipped triangulation becomes regular.)

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References

[1] JESÚS A. DE LOERA, SERKAN HOŞTEN, FRANCISCO SANTOS, AND BERND STURMFELS, The polytope of all triangulations of a point configuration, *Doc. Math.*, **1** (1996) 103–119.

[2] HERBERT EDELSBRUNNER, *Algorithms in combinatorial geometry*, Springer-Verlag, Berlin, 1987.

[3] HERBERT EDELSBRUNNER, An acyclicity theorem for cell complexes in d dimension, *Combinatorica*, **10** (1990) 251–260.

[4] ISRAEL M. GELFAND, MIKHAIL M. KAPRANOV, AND ANDREI V. ZELEVINSKY, *Discriminants, resultants and multidimensional determinants*, Birkhäuser, Boston, 1994.

[5] ISRAEL M. GEL'FAND, ANDREI V. ZELEVINSKIĬ, AND MIKHAIL M. KAPRANOV, Newton polyhedra of principal A-determinants, *Soviet Math. Dokl.*, **40** (1990) 278–281.

[6] HIDEFUMI OHSUGI AND TAKAYUKI HIBI, A normal $(0,1)$-polytope none of whose regular triangulations is unimodular, *Discrete Comput. Geom.*, **21** (1999) 201–204.

[7] FRANCISCO SANTOS, A point configuration whose space of triangulations is disconnected, *J. Amer. Math. Soc.*, **13** (2000) 611–637.

[8] BERND STURMFELS, Gröbner bases of toric varieties, *Tôhoku Math. J.*, **43** (1991) 249–261.

[9] BERND STURMFELS, *Gröbner bases and convex polytopes*, American Mathematical Society, Providence, RI, 1996.

[10] GÜNTER M. ZIEGLER, *Lectures on polytopes*, Springer-Verlag, New York, 1995.